SKEW PRODUCTS OF BERNOULLI SHIFTS WITH ROTATIONS

BY

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ABSTRACTS

This paper investigates certain skew products of Bernoulli shifts with rotations or permutations, and shows that these transformations, if weak mixing, are also Bernoulli.

Sinai [7] and later Ornstein [3] have shown that a mixing transformation of positive entropy has a Bernoulli factor of full entropy. One can make use of Rohlin's theory of measurable partitions [6] to reformulate this result as follows: A mixing transformation of positive entropy h is the skew product of a Bernoulli shift of entropy h with a family $\mathfrak{F}$ of transformations. In light of Ornstein's recent results [4] on K-automorphisms which are not Bernoulli many questions arise as to what kinds of skew products give Bernoulli shifts. In this paper we report results obtained under very restrictive hypotheses, namely that the family $\mathfrak{F}$ consists of two rotations or two permutations and the skew product is measurable with respect to the independent generator of the two-shift. In these cases weak mixing implies Bernoulli.

Preliminaries

Let $(X, B(X), \mu)$ be a Lebesgue space with τ an invertible measure preserving transformation. A partition P is an ordered finite disjoint collection of measurable sets whose union is X. If P and Q are partitions then their join is defined as

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$$P \vee Q = \{P \cap Q_j \,|\, P_i \in P, Q_j \in Q\}$$

ordered lexicographically. A partition P is said to be a *generator* for τ if the sigma-field $B(X)$ is generated by the family of sets $\cup \{\tau^n P : n = 0, \pm 1, \pm 2, \cdots\}$; or equivalently, if there exists a set N, $\mu(N) = 0$, such that for $x, y \notin N$, $\tau^n x$ and $\tau^n y$ belong to different sets of P for some n. Two partitions P and Q are called *independent* $(P \perp Q)$ if

$$\mu(P_i \cap Q_j) = \mu(P_i)\mu(Q_j)$$

for $P_i \in P$, $Q_j \in Q$. A partition P is called an *independent generator* for τ if P is a generator and if

$$P \perp \bigvee_{-m}^{-1} \tau^i P$$

for all $m \geqq 1$. A transformation is called a *Bernoulli shift* if it has an independent generator. A partition P is called *Markov with respect to* τ if

$$\mu\left(\bigcap_{m=0}^{n} \tau^{-m} A_m\right)\mu(A_1) = \mu(A_0 \cap \tau^{-1} A_1)\mu\left(\bigcap_{m=1}^{n} \tau^{-m} A_m\right)$$

for $A_i \in P$ and all positive n.

A transformation is called a *Markov shift* if it has a Markov generator.

THEOREM (Friedman and Ornstein [1]). *A weakly mixing Markov shift is Bernoulli.*

If $P = \{P_1, \cdots, P_N\}$ is a partition we denote by $d(P)$ the vector $d(P) = (\mu(P_1), \cdots \mu(P_N))$. If A is a set of positive measure we denote by $P/A = \{P_1 \cap A, \cdots, P_N \cap A\}$ the partition of A by P and by $d(P/A)$ the vector $d(P/A) = (\mu(P_1 \cap A)/\mu(A), \cdots, \mu(P_N \cap A)/\mu(A))$. If P and Q are partitions of X each with N sets then we define $D(P,Q)$ by

$$D(P,Q) = \sum_{i=1}^{N} \mu(P_i - Q_i) + \mu(Q_i - P_i).$$

If $P^1, P^2, \cdots, P^n$ and $Q^1, Q^2, \cdots, Q^n$ are two sequences of partitions all with N sets then $\bar{d}(\{P^i\}_1^n, \{Q^i\}_1^n)$ is defined by

$$\bar{d}(\{P^i\}_1^n, \{Q^i\}_1^n) = \inf \frac{1}{n} \sum_{i=1}^{n} D(\bar{P}^i, \bar{Q}^i)$$

where the inf is taken over all sequences $\bar{P}^1, \cdots, \bar{P}^n$ and $\bar{Q}^1, \cdots, \bar{Q}^n$ with

$$d\left(\bigvee_{i=1}^{n} P^i\right) = d\left(\bigvee_{i=1}^{n} \bar{P}^i\right)$$

$$d\left(\bigvee_{i=1}^{n} Q^i\right) = d\left(\bigvee_{i=1}^{n} \bar{Q}^i\right)$$

Finally a partition P is said to be *very weak Bernoulli* if for each $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that if $n \geqq 0$ and $m \geqq 1$ then

$$\bar{d}(\{\tau^i P / C\}_0^{n+N}, \{\tau^i P\}_0^{n+N}) < \varepsilon \tag{0}$$

for a collection of sets C in $\bigvee_{-m}^{-1} \tau^i P$ of total measure at least $1 - \varepsilon$.

THEOREM (Ornstein [5]). *If τ has a very weak Bernoulli generator then τ is Bernoulli.*

Skew products

Let $(X, B(X), \mu_X)$ be the bilateral product of the two point sets $\{0,1\}$, each with mass $\frac{1}{2}$ and let σ be the shift, that is,

$$\sigma(x)_n = x_{n+1}, \quad -\infty < n < \infty .$$

Suppose $(Y, B(Y), \mu_Y)$ is a Lebesgue space and ϕ_0 and ϕ_1 are invertible, μ_Y-preserving transformations on Y. Form the direct product

$$(X \times Y, B(X) \times B(Y), \mu_X \times \mu_Y).$$

The transformation

$$\tau(x, y) = (\sigma x, \phi_{x_0}(y))$$

is called the *skew product* of σ with $\{\phi_0, \phi_1\}$, and is an invertible $\mu(= \mu_X \times \mu_Y)$-measure-preserving transformation.

The family $\{\phi_0, \phi_1\}$ is ergodic if $A = \phi_1 A = \phi_0 A$ implies that A has measure 0 or 1. A much more general form of the following lemma can be found in Kakutani [2].

LEMMA 1. *τ is ergodic iff $\{\phi_0, \phi_1\}$ is ergodic.*

A simple extension of this result is obtained by noticing that $\tau \times \tau$ is a skew product of $\sigma \times \sigma$ with $\{\phi_i \times \phi_j \mid i, j \in \{0,1\}\}$. This is

LEMMA 2. *τ is weakly mixing iff $\{\phi_0 \times \phi_0, \phi_0 \times \phi_1, \phi_1 \times \phi_0, \phi_1 \times \phi_1\}$ is ergodic on $Y \times Y$.*

PROOF. τ is weakly if and only if $\tau \times \tau$ is ergodic, hence Lemma 1 applies.

Permutations

We now examine the case when $Y = \{1, 2, \cdots, k\}$, $\mu_Y(i) = 1/k$, $1 \leqq i \leqq k$. In this case ϕ_0 and ϕ_1 are permutations and we have

LEMMA 3. *If $P_{ij} = \{(x,j) \mid x_0 = i\}$, then the partition $P = \{P_{ij} \mid i \in \{0,1\}, j \in Y\}$ is a Markov generator for τ.*

PROOF. If $(x, i) \neq (x', i')$, then either $i \neq i'$, or $i = i'$ and for some n, $x_n \neq x_n'$. In the first case the two points belong to different atoms of P, while in the second case $\tau^n(x, i)$ and $\tau^n(x', i')$ lie in different atoms of P. Thus P is a generator for τ. If $A_0, A_1, \cdots, A_n \in P$, then

$$\mu\left(\bigcap_{m=0}^{n} \tau^{-m} A_m\right) = \frac{1}{2^{n+1}} \cdot \frac{1}{k}$$

if the intersection is not empty, hence the Markovian condition

$$\mu\left(\bigcap_{m=0}^{n} \tau^{-m} A_m\right)\mu(A_1) = \mu(A_0 \cap \tau^{-1} A_1)\mu\left(\bigcap_{m=1}^{n} \bar{\tau}^{m} A_m\right)$$

holds.

Since mixing Markov shifts are Bernoulli we can combine Lemmas 2 and 3 to obtain

THEOREM 1. *If $\{\phi_i \times \phi_j\}$ is ergodic on $Y \times Y$ where Y is a finite set, then τ is isomorphic to a Bernoulli shift.*

For example, if $\phi_0(s) = s$, $\phi_1(s) = s + 1 \pmod{k}$, then τ is a Bernoulli shift. If $k = 4$ and

$$\phi_0(1) = 2,\ \phi_0(2) = 1,\ \phi_0(3) = 4,\ \phi_0(4) = 3,$$

$$\phi_1(1) = 3,\ \phi_1(3) = 1,\ \phi_1(2) = 4,\ \phi_1(4) = 2,$$

then τ is ergodic but not mixing.

Rotations

Now we shall focus on the case where the ϕ_i are rotations. In the remainder of this paper Y will be the unit interval with Lebesgue measure and

$$\phi_0(x) = x + \alpha \pmod 1$$

$$\phi_1(x) = x + \beta \pmod 1$$

where α and β are fixed real numbers. The conditions for weak mixing are given in the following lemma.

LEMMA 4. *τ is weakly mixing iff $\alpha - \beta$ is irrational.*

PROOF. Suppose $\Sigma(a_{n,m})^2 < \infty$,

$$f(y, y') = \Sigma a_{n,m} e^{2\pi i(ny + my')}$$

and for almost all $y, y' \in Y$ and all $\gamma, \gamma' \in \{\alpha, \beta\}$

$$f(y + \gamma, y' + \gamma') = f(y, y').$$

Thus for all n, m and all $\gamma, \gamma' \in \{\alpha, \beta\}$ we have

$$a_{n\cdot m} = a_{n,m} e^{2\pi i(n\gamma + m\gamma')}. \tag{1}$$

If $(n, m) \neq (0,0)$ and $\alpha - \beta$ is irrational this requires $a_{n,m} = 0$, so that τ is weakly mixing. If $\alpha - \beta$ is rational, then (1) has other solutions so $\tau \times \tau$ is not ergodic. This proves Lemma 4.

Our main result in this section is

THEOREM 2. *If $\alpha - \beta$ is irrational, then τ is isomorphic to a Bernoulli shift.*

Our proof will be given after a series of lemmas and can be thought of as a simplified version of the arguments used in Ornstein [5]. We shall make free use of the notation and terminology used in that paper.

For $s \in \{0,1\}$ put

$$P_s = \{x \mid x_0 = s\},\ P_{s0} = P_s \times [0, \tfrac{1}{2}),\ P_{s1} = P_s \times [\tfrac{1}{2}, 1)$$

and

$$Q = \{P_0, P_1\},\ P = \{P_{00}, P_{01}, P_{10}, P_{11}\}.$$

Clearly Q is a generator for σ. We prove

LEMMA 5. *If either α or β s irrational, then P is a generator for τ.*

PROOF. Suppose α is irrational. We can also suppose that for each $x \in X$ there are arbitrarily large k such that k consecutive values of x_n, $n \geqq 0$ are zero. Thus if $(x, y) \neq (x', y')$ and $x = x'$, one can choose n so that the line from (x, y) to (x, y') intersects the line $y = \frac{1}{2}$, that is, $\tau^n(x, y)$ and $\tau^n(x, y')$ lie in different atoms of P. Also if $x \neq x'$, one can choose n so that $\sigma^n x$ and $\sigma^n x'$ lie in different atoms of Q. Thus P is a generator for τ. A similar argument establishes the lemma if β is irrational.

The next three lemmas provide the basic tools for establishing that P is very weak Bernoulli. We assume hereafter that $\alpha - \beta$ is irrational.

LEMMA 6. *If* $C, C' \in \bigvee_{-m}^{-1} \sigma^i Q$ and $\tilde{C} = C \times \{y\}$, $\tilde{C}' = C' \times \{y\}$, *then for all* $n \geqq 0$

$$d\left(\bigvee_0^n \tau^i P / \tilde{C}\right) = d\left(\bigvee_0^n \tau^i P / \tilde{C}\right).$$

PROOF. This follows from the fact that for $s \in \{0,1\}$, and $C \subset P_s$

$$d\left(\bigvee_0^n \tau^i P / \tilde{C}\right) = d\left(\bigvee_0^n \tau^i P / P_s \times \{y\}\right).$$

LEMMA 7. *Given $\varepsilon > 0$, there is a $\delta > 0$ and $N_1 = N_1(\varepsilon) > 0$ such that for $s \in \{0, 1\}$*

$$\bar{d}(\{\tau^i P\}_{i=0}^n / P_s \times \{y\}, \quad \{\tau^i P\}_{i=0}^n / P_s \times \{y'\}) < \varepsilon$$

if $n \geqq N_1$ and $|y - y'| < \delta$.

PROOF. Let $E = X \times [0, \varepsilon/4) \cup X \times [\frac{1}{2}, \frac{1}{2} + \varepsilon/4)$. Apply the ergodic theorem to choose N_1 and F, $\mu(F) < \varepsilon$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_E(\tau^k(x, y)) \leqq \frac{\varepsilon}{2} + \mu(E) \leqq \varepsilon$$

if $n \geqq N_1$ and $(x, y) \notin F$.

If $|\, y - y \,| < \varepsilon/4$ and $(x, y) \notin F$, then for $0 \leqq k \leqq n - 1$, the line from $\tau^k(x, y)$ to $\tau^k(x, y\ )$ does not meet E more than εn times. This proves the lemma.

LEMMA 8. *Suppose k is a positive integer and $Y_i = [i - 1/k, i/k)$, $1 \leqq i \leqq k$.*

Put $Z_i = P_0 \times Y_i$, $1 \leqq i \leqq k$ *and* $Z_i = P_1 \times Y_{i-k}$, *for* $k+1 \leqq i \leqq 2k$. *Given* $\varepsilon' > 0$, *there is an integer* $N_2 = N(\varepsilon')$ *such that if* $C = P_s \times \{y\}$, $s \in \{0,1\}$, *then there is a set* $C' \supseteq C$, $l(C') < \varepsilon$, *such that for each* i, $1 \leqq i \leqq 2k$

$$\left|\ell(T^{N_2}(C - C') \cap Z_i) - \frac{1}{4k}\right| < \varepsilon'$$

where ℓ *is Lebesgue measure on fibers* $X \times \{y\}$.

PROOF. Subdivide each Y_i into intervals Y_{ij} of length $1/kl$, for $1 \leqq j \leqq l$ and form the corresponding portions Z_{ij} of each Z_i. Use weak mixing (Lemma 4) to choose N_2 so that for all i, j, m, and n,

$$|\mu(\tau^{N_2} Z_{ij} \cap Z_{mn}) - \mu(Z_{ij})\mu(Z_{mn})| < \delta.$$

Fix i, j and let W_{ij} be the union of those parts K of Z_{ij} such that $T^{N_2} K \subset Z_{m1} \cap Z_{ml}$ for some m, $1 \leqq m \leqq 2k$. If l is large enough and δ is small enough, then for each m

$$\left|\mu(\tau^{N_2}(Z_{ij} - W_{ij}) \cap Z_m) - \frac{1}{4k^2 l}\right| < \varepsilon.$$

If $C \subseteq Z_{ij}$, let $C' = C \cap W_{ij}$ and the desired result follows.

LEMMA 9. *Given* $\varepsilon > 0$, *there is an* N *such that*

$$\bar{d}(\{{}^iP\}_{i=0}^{N+n}/\tilde{C}, \{\tau^i P\}_0^{N+n}) < \varepsilon \tag{2}$$

for all n *and any set* $\tilde{C} = C \times \{y\}$ *where* $C \in \bigvee_{-m}^{-1} \sigma^i Q$.

PROOF. This lemma completes the proof that P is very weak Bernoulli for one can now integrate (2) over the fibers $\tilde{C}$ of a past set $\tilde{C} \in \bigvee_{-m}^{-1} \tau^i P$ to obtain the desired inequality (0). To prove Lemma 9, first apply Lemma 6 to replace $\tilde{C}$ by $P_s \times \{y\}$, where $C \subseteq P_s$. Choose δ and N_1 from Lemma 7 for $\varepsilon/2$, then choose k such that $1/k < \delta$. Now choose $N_2 = N(\varepsilon')$ from Lemma 8. From Lemma 7 we have, for $1 \leqq i \leqq k$, $n \geqq N_1$

$$\bar{d}(\{\tau^i P\}_{i=0}^n / W_i, \{\tau^i P\}_0^n / Z_i) \leqq \varepsilon/2$$

where $W_i = \tau^{N_2}(C - C') \cap Z_i$. Hence if ε' is small enough, inequality (2) will hold if $n \geqq N_1 + N_2$. Thus the desired N is $N_1 + N_2$. This proves the lemma and completes the proof of Theorem 2.

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